

Local-duality QCD sum rules for the decay constants of heavy-light mesons

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We discuss the leptonic decay constants of heavy–light mesons by means of Borel QCD sum rules in the local-duality (LD) limit of infinitely large Borel mass parameter(s). In this limit, for an appropriate choice of the invariant structures in the QCD correlation functions, all vacuum-condensate contributions vanish and all nonperturbative effects are contained in only one quantity, the effective threshold. We study properties of the LD effective thresholds in the limits of large heavy-quark mass m_Q and small light-quark mass m_q . In the heavy-quark limit, we clarify the role played by the radiative corrections in the effective threshold for reproducing the pQCD expansion of the decay constants of pseudoscalar and vector mesons. We show that the dependence of the meson decay constants on m_q arises predominantly (at the level of 70–80%) from the calculable m_q -dependence of the perturbative spectral densities. Making use of the lattice QCD results for the decay constants of nonstrange and strange pseudoscalar and vector heavy mesons, we obtain solid predictions for the decay constants of heavy–light mesons as functions of m_q in the range from a few to 100 MeV and evaluate the corresponding strong isospin-breaking effects: $f_{D^+} - f_{D^0} = (0.96 \pm 0.09)$ MeV, $f_{D^{*+}} - f_{D^{*0}} = (1.18 \pm 0.35)$ MeV, $f_{B^0} - f_{B^+} = (1.01 \pm 0.10)$ MeV, $f_{B^{*0}} - f_{B^{*+}} = (0.89 \pm 0.30)$ MeV.

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1. INTRODUCTION

The method of QCD sum rules [1–3], based on the exploitation of Wilson’s operator product expansion (OPE) in the study of properties of individual hadrons, has been extensively applied to the decay constants of heavy mesons. An important finding of these analyses was the observation of the strong sensitivity of the decay constants to the precise values of the input OPE parameters and to the algorithm used for fixing the effective threshold [4]. The latter quantity governs the accuracy of the quark–hadron duality approximation and, to a great extent, determines the numerical prediction for decay constants inferred from QCD sum rules: even if the parameters of the truncated OPE are known with high precision, the decay constants may be predicted with only a limited accuracy; the latter represents their systematic uncertainty. In a series of papers [5], we proposed a new algorithm for fixing the effective threshold within the Borel QCD sum rules which allowed us to obtain realistic estimates of the systematic uncertainties. Our procedure opened the possibility to get predictions for the decay constants with a controlled accuracy [6–8] and thus allowed us to address subtle effects that call for a profound accurate treatment, such as the ratios of the decay constants of heavy vector and pseudoscalar mesons [9] or the strong isospin-breaking effects in the decay constants of heavy mesons [10].

Here, we discuss the application of another variant of QCD sum rules to isospin breaking in the decay constants of heavy pseudoscalar and vector mesons. Our analysis takes advantage of the fact that the OPE provides the analytic dependence of the correlation functions on the quark masses; this allows us to study, e.g., the impact of the light-quark mass on heavy-meson decay constants, thus providing access to the strong isospin-violation effects. The approach we describe below seems quite promising for a study of the dependence of a generic hadron observable on quark masses.

A typical Borel QCD sum rule for the decay constant f_H of a heavy (pseudoscalar or vector) $\bar{Q}q$ meson H of mass M_H , consisting of a heavy quark Q with mass m_Q and a light quark q with mass m_q , has the form

$$f_H^2(M_H^2)^N \exp(-M_H^2\tau) = \int_{(m_Q+m_q)^2}^{s_{\text{eff}}^{(N)}(\tau, m_Q, m_q)} ds \exp(-s\tau) s^N \rho_{\text{pert}}(s, m_Q, m_q, \alpha_s) + \Pi_{\text{power}}^{(N)}(\tau, m_Q, m_q, \langle \bar{q}q \rangle, \dots). \quad (1.1)$$

Here, τ is the Borel parameter and N is an integer that depends on the Lorentz structure in the correlation function chosen for the sum rule and on the number of subtractions in the corresponding dispersion representation. Nonperturbative effects appear on the r.h.s. of (1.1) at two places: as power corrections given in terms of vacuum condensates and in the effective threshold $s_{\text{eff}}^{(N)}(\tau, m_Q, m_q)$. Depending on the chosen value of N , nonperturbative effects are distributed in a different way between power corrections and the effective threshold.

We are interested in the dependence of the decay constants of heavy–light mesons on the quark masses, in particular, in the strong isospin-breaking effects in the decay constants (i.e., the difference between the decay constants of $\bar{Q}d$ and

$\bar{Q}u$ mesons induced by the small mass difference $\delta m = m_d - m_u$). We therefore need to properly take into account all effects depending on the light-quark flavour q in the correlation function of the appropriate $\bar{Q}q$ interpolating currents.

Clearly, the m_q -dependence on the l.h.s. of (1.1) is encoded both in the decay constant f_H and in the meson mass M_H . On the r.h.s., the isospin-breaking (IB) effects come from several sources: the m_q -dependence of $\rho_{\text{pert}}(s, m_Q, m_q, \alpha_s)$, the m_q -dependence of the effective threshold $s_{\text{eff}}(\tau)$, the m_q -dependence of the power corrections, and the flavour dependence of the quark condensates, in particular, of $\langle \bar{q}q \rangle$. In general, all these effects mix together, which renders the goal of isolating the IB effects in f_H a complicated task. A careful analysis has been carried out recently in [10], setting $N = 2$ in the case of pseudoscalar and $N = 1$ in the case of vector mesons.

There is, however, a particular limiting case of the sum rule which simplifies the problem considerably. Let us take the sum rule (1.1) for $N = 0$ — this corresponds to considering the correlation function of mass dimension 2, see next section — and set $\tau = 0$, the so-called local-duality (LD) limit. Then the generic sum rule is reduced to the form

$$f_H^2 = \int_{(m_Q+m_q)^2}^{s_{\text{eff}}(m_Q, m_q)} ds \rho_{\text{pert}}(s, m_Q, m_q, \alpha_s). \quad (1.2)$$

On the l.h.s., the M_H contribution has dropped out, thus opening a direct access to the m_q -dependence of the decay constant f_H . On the r.h.s. of (1.1), all nonperturbative power corrections vanish: the Borelized correlator has mass dimension 2, whereas the lowest-dimensional vacuum condensates (the quark condensate $m_Q \langle \bar{q}q \rangle$ and the gluon condensate $\langle GG \rangle$) have dimension 4. This means that their contribution to the sum rule comes with a power of τ and thus vanishes in the limit $\tau \rightarrow 0$. Noteworthy, all nonperturbative effects enter now through a single quantity — the effective threshold. The functional dependence of s_{eff} on the quark masses, in particular, on m_q , may be determined from the general properties of the decay constants of heavy–light mesons in QCD, whereas the unknown numerical coefficients of this functional dependence may be obtained from a few well-measured values of the decay constants, e.g., for nonstrange and strange heavy mesons. Having at our disposal a parameterization of the LD effective threshold, Eq. (1.2) allows us to study the decay constants of heavy–light pseudoscalar and vector mesons as functions of m_q and to access the strong isospin-breaking effects related to the small difference of the u - and d -quark masses in QCD.

We would like to emphasize that, as will be demonstrated below, the main m_q -dependence of the decay constants originates from the calculable m_q -dependence of the perturbative spectral densities. Therefore, the LD limit opens the possibility of a reliable analysis of the m_q -dependence and the strong IB effects in the decay constants of heavy–light mesons (and, in principle, also in other quantities).

This paper is organized as follows: In Sec. 2, we recall the spectral densities of the QCD correlation functions relevant for our LD sum-rule analysis. In Sec. 3, we study the m_Q - and m_q -dependences of the effective thresholds by making use of an appropriate mass scheme (pole mass and running mass) for the heavy quarks. In Sec. 4, we perform the numerical analysis of the decay constants of heavy–light pseudoscalar and vector mesons and obtain predictions for strong IB effects in the decay constants. Section 5 gives our conclusions. The Appendix collects some details of treating the isospin-breaking effects within the OPE, which, in our opinion, deserve to be presented.

2. LOCAL-DUALITY SUM RULES FOR f_P AND f_V

Let us consider two-point QCD sum-rules for decay constants of pseudoscalar (P) and vector (V) mesons built up of one massive quark Q with mass m_Q and one light quark q with mass m_q . We consider the axial-vector current

$$j_\mu^5(x) = \bar{q}(x) \gamma_\mu \gamma_5 Q(x) \quad (2.1)$$

and the vector current

$$j_\mu(x) = \bar{q}(x) \gamma_\mu Q(x) \quad (2.2)$$

as interpolating currents for the pseudoscalar and vector mesons, respectively. The corresponding correlation functions involve two Lorentz structures, the transverse structure $g_{\mu\nu}p^2 - p_\mu p_\nu$ and the longitudinal structure $p_\mu p_\nu$:

$$\Pi_{\mu\nu}^5(p) = i \int dx \exp(ipx) \langle T(j_\mu^5(x) j_\nu^{5\dagger}(0)) \rangle = (g_{\mu\nu}p^2 - p_\mu p_\nu) \Pi_T^5(p^2) + p_\mu p_\nu \Pi_L^5(p^2), \quad (2.3)$$

$$\Pi_{\mu\nu}(p) = i \int dx \exp(ipx) \langle T(j_\mu(x) j_\nu^\dagger(0)) \rangle = (g_{\mu\nu}p^2 - p_\mu p_\nu) \Pi_T(p^2) + p_\mu p_\nu \Pi_L(p^2). \quad (2.4)$$

For $\Pi_{\mu\nu}^5(p)$, we study the longitudinal structure $p_\mu p_\nu$, as it contains the ground-state pseudoscalar-meson contribution

$$p_\mu p_\nu \frac{f_P^2}{M_P^2 - p^2}, \quad (2.5)$$

with

$$\langle 0 | \bar{q} \gamma_\mu \gamma_5 Q | P(p) \rangle = i f_P p_\mu. \quad (2.6)$$

For $\Pi_{\mu\nu}(p)$, we study the transverse structure $g_{\mu\nu} p^2 - p_\mu p_\nu$, which contains the ground-state vector-meson contribution

$$(g_{\mu\nu} M_V^2 - p_\mu p_\nu) \frac{f_V^2}{M_V^2 - p^2}, \quad (2.7)$$

with

$$\langle 0 | \bar{q} \gamma_\mu Q | V(p) \rangle = M_V f_V \epsilon_\mu(p). \quad (2.8)$$

As already noted, power corrections for dimension-2 correlation functions must vanish in the LD limit $\tau = 0$. We now present their explicit form. The leading power correction to $\Pi_{\mu\nu}^5(p)$ is given by the quark condensate and easily derived:

$$\begin{aligned} \Pi_{\mu\nu}^5(p) |_{\langle \bar{q}q \rangle} = & - (p^2 g_{\mu\nu} - p_\mu p_\nu) m_Q \langle \bar{q}q \rangle \left[\frac{1}{p^2} \left(\frac{1}{m_Q^2 - p^2} + \frac{\frac{1}{2} m_Q m}{(m_Q^2 - p^2)^2} \right) \right] \\ & + p_\mu p_\nu \langle \bar{q}q \rangle \left[\frac{-m_Q}{p^2} \left(\frac{1}{m_Q^2 - p^2} + \frac{\frac{1}{2} m_Q m}{(m_Q^2 - p^2)^2} \right) + \frac{m}{(m_Q^2 - p^2)^2} \right]. \end{aligned} \quad (2.9)$$

The Borel transform $p^2 \rightarrow \tau$ [defined such that $\frac{1}{a-p^2} \rightarrow \exp(-a\tau)$] of the longitudinal part of $\Pi_{\mu\nu}^5(p, q=0) |_{\langle \bar{q}q \rangle}$ reads

$$p_\mu p_\nu \frac{\langle \bar{q}q \rangle}{2m_Q^2} \left[-(2m_Q + m) (\exp(-m_Q^2 \tau) - 1) + m m_Q^2 \tau \exp(-m_Q^2 \tau) \right]. \quad (2.10)$$

Obviously, this expression vanishes in the LD limit of interest, $\tau = 0$. By changing the sign of the light-quark mass, the power corrections for the vector correlator $\Pi_{\mu\nu}(p, m_Q, m_q)$ are easily found from $\Pi_{\mu\nu}^5(p, m_Q, m_q)$: $\Pi_{\mu\nu}^{\text{power}}(p, m_Q, m_q) = \Pi_{\mu\nu}^{5,\text{power}}(p, m_Q, -m_q)$. Also the Borelized power corrections for the vector correlator vanish in the limit $\tau = 0$.

Since the power corrections do not contribute to the LD sum rule under consideration, we need to consider only the perturbative contributions. After applying the duality cuts at s_{eff} , separately in the pseudoscalar and the vector channels, performing the Borel transform $p^2 \rightarrow \tau$, and setting $\tau \rightarrow 0$, the corresponding sum rules take the form

$$f_{P,V}^2 = \int_{(m_Q+m_q)^2}^{s_{\text{eff}}} ds \rho_{P,V}^{\text{pert}}(s, m_Q, m_q). \quad (2.11)$$

The functions $\rho_P(s, m_Q, m_q)$ and $\rho_V(s, m_Q, m_q)$ in (2.11) are the spectral densities of the invariant functions $\Pi_L^5(p^2)$ and $\Pi_T(p^2)$, respectively.

Let us emphasize that in (2.11) both the full spectral densities and the decay constants are scale-independent quantities. Therefore, the effective thresholds are scale-independent objects, too. In perturbation theory, the spectral densities are calculated as power expansions in $a \equiv \alpha_s(\mu)/\pi$, $\alpha_s(\mu)$ the strong coupling in the $\overline{\text{MS}}$ -scheme at scale μ :

$$\rho_{P,V}^{\text{pert}}(s, m_Q, m_q) = \rho_{P,V}^{(0)}(s, m_Q, m_q) + a \rho_{P,V}^{(1)}(s, m_Q, m_q) + a^2 \rho_{P,V}^{(2)}(s, m_Q, m_q) + O(a^3). \quad (2.12)$$

In practice, one adopts truncated expansions of the spectral densities; this leads to a scale-dependence of the spectral densities. As the result, the effective thresholds will also depend on the scale, to compensate the scale-dependence of the spectral densities emerging in the course of truncation. Explicitly, the leading-order (LO) spectral densities read

$$\rho_P(s, m_Q, m) = \frac{N_c}{8\pi^2} (m_Q + m_q)^2 (s - (m_Q - m_q)^2) \frac{\lambda^{1/2}(s, m_Q^2, m_q^2)}{s^3} \theta(s - (m_Q + m_q)^2), \quad (2.13)$$

$$\rho_V(s, m_Q, m) = \frac{N_c}{24\pi^2} (2s + (m_Q + m_q)^2) (s - (m_Q - m_q)^2) \frac{\lambda^{1/2}(s, m_Q^2, m_q^2)}{s^3} \theta(s - (m_Q + m_q)^2). \quad (2.14)$$

Obviously, the lower integration limit in (2.11) is determined by the threshold in the correlation functions.

In Eq. (2.12), we may employ different definitions of the quark masses: The most advanced calculation of the pseudoscalar and vector spectral densities including order- $O(a^2)$ terms was performed [11], for a massless light quark, in terms of the heavy-quark pole mass. The expansion in terms of the heavy-quark pole mass is appropriate for considering the heavy-quark limit, which we address in Sec. 3 A.

However, the pole-mass expansion leads to a rather slow convergence of the perturbative expansion for the decay constants [6, 7, 12]. The convergence improves considerably when one rearranges the perturbative expansion in terms of the running $\overline{\text{MS}}$ masses. Therefore, for the practical analysis of the m_q -dependences of the meson decay constants in Sec. 4, we make use of the perturbative expansion in terms of the running $\overline{\text{MS}}$ masses of the light and the heavy quarks. The corresponding NLO and NNLO functions $\rho_P^{(i)}$ ($i = 1, 2$) in (2.12) necessary for such an analysis are found from the spectral densities of the pseudoscalar correlation function given in [12] by multiplying them by $1/s^2$. Similarly, the transverse spectral densities $\rho_V^{(i)}$ in (2.12) are found from the spectral densities of [13] by multiplying them by $1/s$. In our analysis, we make use of the exact LO perturbative spectral density given by (2.13), at the NLO we keep the terms $O(am_q^0)$ and $O(am_q^1)$, and in the NNLO we keep the only known terms of order $O(a^2 m_q^0)$.

We would like to emphasize that the perturbative spectral density (2.12) does not generate terms of order $m_q \log(m_q)$ in the dual correlator (2.11). This observation will be crucial for discussing properties of the effective thresholds in the next section.

3. DEPENDENCE OF THE EFFECTIVE THRESHOLDS ON THE QUARK MASSES

Let us now consider the dependences of the effective threshold on the quark masses m_Q and m_q .

A. Heavy-quark limit in the pole-mass scheme

We start with the heavy-quark limit of the decay constants and, for the sake of argument, consider first a massless light quark: $m_q = 0$. We can make use of any scheme for the heavy-quark mass, but we start with the pole-mass scheme, which leads to a more transparent behaviour of the effective threshold. We first isolate the pole mass, which we denote M_Q , in the effective threshold:

$$\sqrt{s_{\text{eff}}} = M_Q + z_{\text{eff}}^{\text{pole}}(M_Q). \quad (3.1)$$

For the decay constant of pseudoscalar and vector $\bar{Q}q$ mesons, using results from [11], we obtain in the limit $M_Q \rightarrow \infty$

$$\begin{aligned} f_P^2 M_Q &= \frac{N_c}{3\pi^2} (z_{\text{eff}}^{\text{pole}})^3 \left[1 + \bar{a} \frac{C_F}{12} \left\{ 45 + 4\pi^2 + 18 \log \left(M_Q / 2z_{\text{eff}}^{\text{pole}} \right) \right\} + O(\bar{a}^2) \right], \\ f_V^2 M_Q &= \frac{N_c}{3\pi^2} (z_{\text{eff}}^{\text{pole}})^3 \left[1 + \bar{a} \frac{C_F}{12} \left\{ 33 + 4\pi^2 + 18 \log \left(M_Q / 2z_{\text{eff}}^{\text{pole}} \right) \right\} + O(\bar{a}^2) \right]. \end{aligned} \quad (3.2)$$

Hereafter, we denote $\bar{a} \equiv \bar{\alpha}_s(M_Q)/(4\pi)$, $\bar{\alpha}_s(M_Q)$ the running strong coupling in the $\overline{\text{MS}}$ scheme at scale M_Q , and use the standard notations $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $T = 1/2$, and n_l the number of massless quarks [11].

Since only the near-threshold behaviour of the spectral densities is relevant for the leading behaviour in the large- M_Q limit, we may obtain also the $O(\bar{a}^2)$ terms in the dual correlation functions (i.e., the r.h.s. of (3.2)) from the analytical expressions for these spectral densities given by Eqs. (30) and (31) of [11].

In the limit $M_Q \rightarrow \infty$, the dual correlation functions, expressed in terms of $z_{\text{eff}}^{\text{pole}}(M_Q)$, do not contain corrections of order $\bar{a}^n M_Q$ (this property will not hold in the running-mass scheme) but still contain $\log(M_Q)$ terms of the type $\left(\bar{a} \log \left(M_Q / z_{\text{eff}}^{\text{pole}} \right) \right)^n$, $\bar{a} \left(\bar{a} \log \left(M_Q / z_{\text{eff}}^{\text{pole}} \right) \right)^{n-1}$, etc. The terms $\left(\bar{a} \log \left(M_Q / z_{\text{eff}}^{\text{pole}} \right) \right)^n$, although formally of order \bar{a}^n , remain unsuppressed in the limit $M_Q \rightarrow \infty$. To treat all terms containing $\log(M_Q)$, it is important to emphasize that they are exactly the same in the vector and the pseudoscalar sum rules. Therefore, they may be resummed by introducing a properly defined effective threshold $z_{\text{eff}}^{\text{HQ}}$, one and the same in the pseudoscalar and the vector channels.

The explicit relation between $z_{\text{eff}}^{\text{pole}}(M_Q)$ and $z_{\text{eff}}^{\text{HQ}}$, including also the a^2 terms, reads:

$$z_{\text{eff}}^{\text{pole}}(M_Q) = z_{\text{eff}}^{\text{HQ}} \left[1 - \bar{a} d_{11} \log \left(M_Q / z_{\text{eff}}^{\text{HQ}} \right) - \bar{a}^2 d_{22} \left(\log \left(M_Q / z_{\text{eff}}^{\text{HQ}} \right) \right)^2 - \bar{a}^2 d_{21} \left(M_Q / z_{\text{eff}}^{\text{HQ}} \right) + O(\bar{a}^3) \right], \quad (3.3)$$

$$d_{11} = \frac{C_F}{2},$$

$$d_{22} = \frac{C_F}{24} [11C_A - 3C_F - 4n_l T],$$

$$d_{21} = \frac{C_F}{432} [C_F(63 + 48\pi^2) + C_A(1401 + 76\pi^2 - 396 \log 2) - 4n_l T(129 + 8\pi^2 - 36 \log 2)].$$

The new quantity $z_{\text{eff}}^{\text{HQ}}$, which has the meaning of the effective threshold in HQET, absorbs all $\log(M_Q)$ terms on the r.h.s. of the sum rules (3.2); the latter assume a form in which the HQ limit may be easily taken:

$$f_P^2 M_Q = \frac{N_c}{3\pi^2} (z_{\text{eff}}^{\text{HQ}})^3 \left[1 + \bar{a} \frac{C_F}{12} \{45 + 4\pi^2 - 18 \log 2\} + O(\bar{a}^2) \right],$$

$$f_V^2 M_Q = \frac{N_c}{3\pi^2} (z_{\text{eff}}^{\text{HQ}})^3 \left[1 + \bar{a} \frac{C_F}{12} \{33 + 4\pi^2 - 18 \log 2\} + O(\bar{a}^2) \right]. \quad (3.4)$$

We did also calculate the $O(\bar{a}^2)$ contributions but do not present their explicit expressions here. The expressions (3.4) immediately lead to the ratio of the decay constants in the heavy-quark limit [14]. Including also $O(\bar{a}^2)$ corrections, we obtain¹

$$\frac{f_V}{f_P} = 1 - \bar{a} \frac{C_F}{2} + \bar{a}^2 \frac{C_F}{144} \left\{ 93C_F + 4(-41 + 19n_l)T + C_A(-263 - 24\pi^2(\log 2 - 1)) \right. \\ \left. + 16\pi^2(T + C_F(\log 8 - 4)) + 36(C_A - 2C_F)\zeta_3 \right\} + O(\bar{a}^3), \quad (3.5)$$

with $\zeta_3 \simeq 1.202$. Notice that for finite M_Q , $z_{\text{eff}}^{\text{pole}}$ contains not only the logarithmic corrections, which are the same in the pseudoscalar and the vector channels, but also the $1/M_Q$ corrections,

$$z_{\text{eff}}^{\text{pole}} = z_{\text{eff}}^{\text{HQ}} \left[1 - \bar{a} \frac{C_F}{2} \log \left(M_Q / z_{\text{eff}}^{\text{HQ}} \right) + O(\bar{a}^2) \right] + O(1/M_Q), \quad (3.6)$$

which are different for the thresholds in the pseudoscalar and the vector sum rules.

B. Combined heavy-quark and chiral limits in the pole-mass scheme

The results (3.2) are obtained for a massless light quark. Switching on a small light-quark mass m_q , the leading corrections generated by integration of the perturbative spectral densities are proportional to m_q ; as already noticed above, no chiral logs of the kind $m_q \log(m_q)$ arise from integrating the spectral densities. Therefore, chiral logs in the decay constants may be generated only by chiral logs in the effective threshold. Moreover, in order to study the chiral logs in the decay constants, it is sufficient to make use of the perturbative spectral densities for $m_q = 0$. On the other hand, heavy-meson chiral perturbation theory (HMChPT) [15], requires the appearance of chiral logs, which we denote as z_L^{HQ} in the chiral expansion of the decay constants in the heavy-quark limit. Since the only source of such terms is the effective threshold, we write

$$z_{\text{eff}}^{\text{HQ}} = z_0^{\text{HQ}} \left(1 + z_L^{\text{HQ}} \right) + \dots, \quad (3.7)$$

where the dots denote linear and higher-order terms in the light-quark mass m_q . The coefficient z_L^{HQ} can now be fixed by matching to HMChPT [15], which provides the explicit chiral logs $R_\chi(m_q)$ in the ratio $f_{H_q}(m_q)/f_{H_{ud}}$, with H_{ud} a heavy meson with a light valence quark of the average mass $m_{ud} \equiv (m_u + m_d)/2$. Finally, we obtain

$$z_L^{\text{HQ}}(m_q) = [1 + R_\chi(m_q)]^{2/3} - 1 \approx \frac{2}{3} R_\chi(m_q). \quad (3.8)$$

The explicit expression for $R_\chi(m_q)$ was derived in [15] and presented by Eq. (A.3) of [10].

¹ The second-order pseudoscalar and vector spectral densities near the threshold, Eqs. (30) and (31) in [11], contain three unknown constants, \tilde{c}_{FF} , \tilde{c}_{FA} , and \tilde{c}_{FL} , which cancel in the ratio f_V/f_P .

C. Quark-mass dependences of the effective threshold in the running-mass scheme

For practical sum-rule analyses of decay constants, one prefers the $\overline{\text{MS}}$ running-mass scheme since it entails a better convergence of the perturbative expansion [6, 7, 12]. It is not difficult to perform the limit $\overline{m}_Q(\mu) \rightarrow \infty$ also for the running-mass correlation function. Also therein one can write

$$\sqrt{s_{\text{eff}}} = \overline{m}_Q(\mu) + \bar{z}_{\text{eff}}(\mu). \quad (3.9)$$

The effective threshold $\bar{z}_{\text{eff}}(\mu)$ in the $\overline{\text{MS}}$ scheme is related to $z_{\text{eff}}^{\text{pole}}$ introduced in the pole-mass scheme through an obvious relation which just expresses the fact that the upper integration limit s_{eff} is a scheme-independent quantity:

$$M_Q + z_{\text{eff}}^{\text{pole}} = \overline{m}_Q(\mu) + \bar{z}_{\text{eff}}(\mu). \quad (3.10)$$

In particular, for $\mu = \overline{m}_Q$, taking into account that

$$M_Q = \overline{m}_Q (1 + C_F \bar{a}), \quad \overline{m}_Q \equiv \overline{m}_Q(\overline{m}_Q), \quad (3.11)$$

one finds

$$\bar{z}_{\text{eff}}(\overline{m}_Q) = z_{\text{eff}}^{\text{pole}} + C_F \bar{a} \overline{m}_Q + O(\bar{a}^2). \quad (3.12)$$

Since $z_{\text{eff}}^{\text{pole}}$ does not contain terms scaling as m_Q in the limit $m_Q \rightarrow \infty$, $\bar{z}_{\text{eff}}(\mu)$ should contain terms which diverge as powers of $a^n M_Q$ in this limit. This is, of course, no obstacle for using $\bar{z}_{\text{eff}}(\mu)$ in the analysis of the decay constants of charmed or beauty mesons but makes this quantity not particularly convenient for studying the heavy-quark limit of the sum rules. The terms in $\bar{z}_{\text{eff}}(\mu)$ divergent as $m_Q \rightarrow \infty$, however, do not lead to divergent terms in the decay constants; also, the behaviour of the spectral densities in the $\overline{\text{MS}}$ scheme is a bit more tricky than in the pole-mass scheme. The dual correlator is determined by the end-point behaviour of the spectral densities; as already mentioned in [12], the higher-order spectral densities in the $\overline{\text{MS}}$ scheme do not vanish at the threshold but behave as $v^{2-k} \alpha_s \log(v)^k$, $v = \frac{1-s/M_Q^2}{1+s/M_Q^2}$. Finally, when the $\overline{\text{MS}}$ spectral densities are used and the duality cut is expressed via $z_{\text{eff}}^{\text{HQ}}$, all terms containing powers of \overline{m}_Q — those coming from the integrals of the spectral densities and those contained in $z_{\text{eff}}(\overline{m}_Q)$ — cancel each other, yielding a sum rule for f_H^2 that can be also obtained just by expressing M_Q via \overline{m}_Q in (3.2), e.g.,

$$f_P^2 \overline{m}_Q = \frac{N_c}{3\pi^2} \left(z_{\text{eff}}^{\text{HQ}} \right)^3 \left[1 + \frac{1}{12} \bar{a} C_F (33 + 4\pi^2) \right]. \quad (3.13)$$

Let us now switch on a small light-quark mass m_q . The spectral densities are now treated as functions of $\overline{m}_Q(\mu)$ and $\overline{m}_q(\mu)$. Taking into account that the effective threshold depends on the scale μ only because of the truncation of the perturbative series, and that the chiral logs have been fixed in the pole-mass scheme, it is convenient to work with the following parameterization for s_{eff} :

$$\sqrt{s_{\text{eff}}} = M_Q^{(2)} + z_{\text{eff}}^{\text{pole}} (1 + z_L^{\text{HQ}}) + \overline{m}_q(\mu) + \bar{z}'_1(\mu) \overline{m}_q(\mu) + O(\overline{m}_q^2). \quad (3.14)$$

The pole mass $M_Q^{(2)}$ here is understood as being expressed via the running mass $\overline{m}_Q(\mu)$ (e.g., [16]) at $O(a^2)$ accuracy, the available accuracy of the correlation function. We can rewrite this expression in a form similar to (3.9) in terms of $\bar{z}_0(\mu) = z_{\text{eff}}^{\text{pole}} + \delta \overline{m}_Q(\mu)$, where $\delta \overline{m}_Q(\mu) \equiv M_Q^{(2)} - \overline{m}_Q(\mu)$:

$$\sqrt{s_{\text{eff}}} = \overline{m}_Q(\mu) + \overline{m}_q(\mu) + \bar{z}_0(\mu) \left(1 + \frac{\bar{z}_0(\mu) - \delta \overline{m}_Q(\mu)}{\bar{z}_0(\mu)} z_1^L + \bar{z}_1(\mu) \overline{m}_q(\mu) \right) + O(\overline{m}_q^2). \quad (3.15)$$

Let us recall that the chiral logs z_L^{HQ} have been calculated in the heavy-quark limit; at finite values of m_Q , chiral logs receive corrections which are unknown. So we take into account only the known leading effect of chiral logs, to study whether or not their impact on the isospin breaking is crucial. Two other parameters of the effective threshold — $z_{\text{eff}}^{\text{pole}}$ and $\bar{z}'_1(\mu)$ if one makes use of the parameterization (3.14), or $\bar{z}_0(\mu)$ and $\bar{z}_1(\mu)$ if one works with (3.15) — are unknown and will be fixed by using some external benchmark results for the decay constants from lattice QCD. The inclusion of higher-order terms in the light-quark mass has no impact on the decay constants; thus, such terms are not considered.

4. NUMERICAL ANALYSIS OF THE SUM RULES

Now, we turn to the numerical estimates. For the relevant OPE parameters, we adopt the following numerical input:

$$\begin{aligned}
(\overline{m}_d - \overline{m}_u)(2 \text{ GeV}) &= (2.67 \pm 0.22) \text{ MeV} [17], & \overline{m}_{ud}(2 \text{ GeV}) &\equiv \frac{\overline{m}_u + \overline{m}_d}{2} = (3.70 \pm 0.17) \text{ MeV} [17], \\
\overline{m}_s(2 \text{ GeV}) &= (93.9 \pm 1.1) \text{ MeV} [17], \\
\overline{m}_c(\overline{m}_c) &= (1.275 \pm 0.025) \text{ GeV} [18], & \overline{m}_b(\overline{m}_b) &= (4.247 \pm 0.034) \text{ GeV} [7], \\
\alpha_s(M_Z) &= 0.1182 \pm 0.0012 [17].
\end{aligned} \tag{4.1}$$

We have checked that employing slightly different values of the quark masses (compatible within uncertainties with those in Eq. (4.1)) which have been reported in the lattice analyses of pseudoscalar mesons [$\overline{m}_b(\overline{m}_b) = (4.190 \pm 0.021) \text{ GeV}$, $\overline{m}_c(\overline{m}_c) = (1.286 \pm 0.030) [17]$] or vector mesons [$\overline{m}_b(\overline{m}_b) = (4.26 \pm 0.10) \text{ GeV} [19]$, $\overline{m}_c(\overline{m}_c) = (1.348 \pm 0.046) \text{ GeV} [20]$, $\overline{m}_s(\overline{m}_s) = (99.6 \pm 4.3) \text{ MeV} [20]$] do not affect our numerical estimates for the IB within the quoted uncertainties.

We work with the effective threshold in the form (3.15) and consider the following three Ansätze for this quantity:

1. “Constant” threshold: the $\bar{z}_1(\mu)$ term in the effective threshold (3.15) and the chiral logs z_L^{HQ} are neglected; the only unknown parameter $\bar{z}_0(\mu)$ is fixed from the lattice results for the decay constants of the isospin-symmetric heavy mesons, with $m_q = m_{ud}$.
2. “Linear” threshold: the chiral logs z_L^{HQ} are neglected and the parameters $\bar{z}_0(\mu)$ and $\bar{z}_1(\mu)$ are fixed by the lattice QCD results for the decay constants at two m_q values, for the isospin-symmetric and the strange heavy mesons.
3. “Linear + log” threshold: the known leading chiral logs represented by z_L^{HQ} are included; the parameters $\bar{z}_0(\mu)$ and $\bar{z}_1(\mu)$ are fixed from the lattice QCD results for the decay constants at two m_q values, for isospin-symmetric and strange heavy mesons.

As we have already noticed, because of the truncation of the perturbative expansion, the truncated spectral densities depend on the scale μ . Obviously, the parameters \bar{z}_0 and \bar{z}_1 are also μ -dependent.

For fixing the parameters of the effective thresholds, we make use of the following results from lattice QCD:

$$\begin{aligned}
f_D &= (212.15 \pm 1.45) \text{ MeV}, & f_{D_s}/f_D &= 1.1716 \pm 0.0032 [17], \\
f_{D^*} &= (223.5 \pm 8.3) \text{ MeV}, & f_{D_s^*}/f_{D^*} &= 1.203 \pm 0.054 [21], \\
f_B &= (186.0 \pm 4.0) \text{ MeV}, & f_{B_s}/f_B &= 1.205 \pm 0.007 [17], \\
f_{B^*} &= (186.4 \pm 7.1) \text{ MeV}, & f_{B_s^*}/f_{B^*} &= 1.197 \pm 0.055 [21].
\end{aligned} \tag{4.2}$$

In these formulas, f_H denotes the decay constant of the isospin-averaged heavy–light mesons with the light-quark mass m_{ud} , whereas f_{H_s} denotes the decay constant of the heavy strange mesons.

Table 1 summarizes the obtained effective thresholds corresponding to our three ansätze and presents estimates of the strong IB. For our final estimates, we perform a bootstrap analysis of the uncertainties assuming that the OPE

Table 1: Parameters of the effective thresholds and resulting IB in the decay constants of heavy pseudoscalar and vector mesons. The parameter z_L in the effective threshold for the “linear + log” ansatz is fixed by ChHQET in the heavy-quark limit.

Meson	Threshold	z_0 [GeV]	z_1 [GeV ⁻¹]	$f_{M_d} - f_{M_u}$ [MeV]
D	Constant	1.363 ± 0.213		1.222 ± 0.219
	Linear	1.366 ± 0.203	-0.365 ± 0.301	1.050 ± 0.102
	Linear + log	1.225 ± 0.194	-1.422 ± 0.304	0.929 ± 0.088
D^*	Constant	1.207 ± 0.147		1.276 ± 0.217
	Linear	1.207 ± 0.138	0.006 ± 0.464	1.281 ± 0.389
	Linear + log	1.087 ± 0.139	-0.978 ± 0.524	1.080 ± 0.381
B	Constant	1.501 ± 0.143		0.792 ± 0.081
	Linear	1.499 ± 0.134	0.498 ± 0.076	1.113 ± 0.108
	Linear + log	1.365 ± 0.136	-0.639 ± 0.147	0.918 ± 0.091
B^*	Constant	1.534 ± 0.163		0.839 ± 0.076
	Linear	1.533 ± 0.152	0.227 ± 0.401	1.010 ± 0.317
	Linear + log	1.395 ± 0.152	-0.938 ± 0.448	0.786 ± 0.311

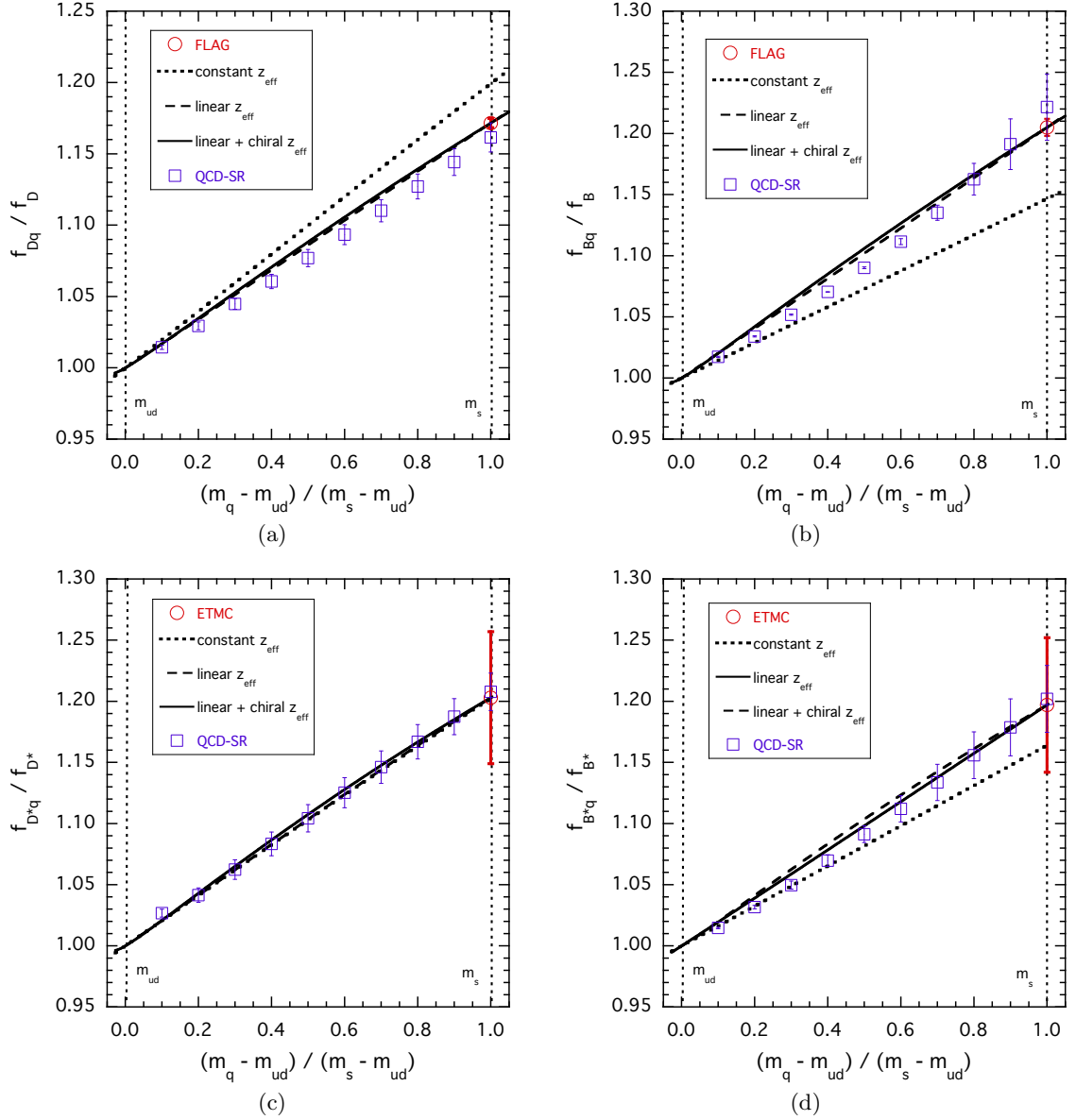


Fig. 1: Dependence of the ratio $f(m_q)/f(m_{ud})$ for pseudoscalar $\bar{c}q$ (a) and $\bar{b}q$ (b) mesons, and vector $\bar{c}q$ (c) and $\bar{b}q$ (d) mesons. Dotted lines correspond to the constant (m_q -independent) effective threshold [ansatz (1)] fixed from the known lattice QCD results for the decay constants of the D , B , D^* and B^* mesons. Dashed lines correspond to the effective threshold linear in m_q [ansatz (2)], the parameters of which are fixed by the lattice results for strange heavy mesons D_s , B_s , D_s^* and B_s^* . Solid lines correspond to the effective threshold containing the known chiral logs in addition to the function linear in m_q [ansatz (3)]. In all cases, the results for the central values of the threshold parameters in Table 1 are displayed. We also show results from an alternative analysis based on Borel QCD sum rules [10].

parameters in (4.1) have Gaussian distributions with corresponding Gaussian errors, whereas the scale μ has a flat distribution in the range $1 < \mu$ (GeV) < 3 for charmed mesons and $3 < \mu$ (GeV) < 6 for beauty mesons.

As soon as the effective thresholds are known, we readily get the decay constants f_{H_q} as a function of the scale-independent ratio $(\overline{m}_q - \overline{m}_{ud})/(\overline{m}_s - \overline{m}_{ud})$. The results for the ratios of the decay constants $f_{H_q}/f_{H_{ud}}$ are shown in Figure 1.

Notice that the results corresponding to a constant effective threshold [ansatz (1)] are quite close to those obtained including the m_q -dependence [ansatz (2)]. So, an important conclusion to be drawn from our results is that the details of the *unknown* m_q -dependence of the effective thresholds are not crucial for the m_q -dependence of the decay constants and for the IB: the latter are determined to large extent by the *known* m_q -dependence of the spectral densities and can thus be reliably controlled in our approach.

5. SUMMARY AND CONCLUSIONS

We addressed the local-duality (LD) limit, $\tau = 0$, of the Borel QCD sum rules for the decay constants of heavy-light pseudoscalar and vector mesons. An invaluable feature of the LD limit is that for a proper choice of the correlation function, all vacuum-condensate contributions vanish and the full nonperturbative QCD dynamics is parameterized in terms of merely one quantity — the effective threshold. Our analysis demonstrates that the effective threshold has a nontrivial functional dependence on the masses of the heavy and the light quarks, m_Q and m_q , respectively. This dependence has been parameterized in the form suggested by the behaviour of the decay constants in the known limits: the chiral limit for m_q and the heavy-quark limit for m_Q . In the heavy-quark limit, we clarify the role played by the radiative corrections in the effective threshold for reproducing the pQCD expansion of the decay constants of pseudoscalar and vector mesons.

This paper elucidates the dependence of the decay constants on a light-quark mass m_q in the range $m_{ud} < m_q < m_s$. Fixing a few numerical parameters of the effective threshold by using the available accurate inputs from lattice QCD, we have derived the full analytic dependence of the decay constants $f_H(m_q)$ on the light-quark mass m_q . The resulting dependence of the decay constants $f_H(m_q)$ on m_q emerges from two sources: (i) from the m_q -dependence of the QCD perturbative spectral densities known explicitly as expansion in powers of α_s and (ii) from the m_q -dependence of the effective threshold known approximately. An important outcome of our analysis is that the variation of the decay constants with respect to m_q comes to a great extent (70–80% of the full effect) comes from the rigorously calculable dependence on m_q of the perturbative spectral densities and is therefore under a good theoretical control.

As our final estimates of the IB, we take the average of the results corresponding to the linear and the linear + log effective thresholds in Table 1:

$$\begin{aligned} f_{D^+} - f_{D^0} &= (0.96 \pm 0.09) \text{ MeV} , \\ f_{D^{*+}} - f_{D^{*0}} &= (1.18 \pm 0.35) \text{ MeV} , \\ f_{B^0} - f_{B^+} &= (1.01 \pm 0.10) \text{ MeV} , \\ f_{B^{*0}} - f_{B^{*+}} &= (0.89 \pm 0.30) \text{ MeV} . \end{aligned} \tag{5.1}$$

Sizeably larger uncertainties of the IB in the decay constants of vector mesons compared to pseudoscalar mesons are related to larger uncertainties of the input lattice QCD results for the corresponding ratios $f_{H_s}/f_{H_{ud}}$.

These estimates are in good agreement with the results of our recent analysis within a different version of QCD sum rules — the Borel sum rules with τ -dependent threshold [10]:

$$\begin{aligned} f_{D^+} - f_{D^0} &= (0.97 \pm 0.13) \text{ MeV} , \\ f_{D^{*+}} - f_{D^{*0}} &= (1.73 \pm 0.27) \text{ MeV} , \\ f_{B^0} - f_{B^+} &= (0.90 \pm 0.13) \text{ MeV} , \\ f_{B^{*0}} - f_{B^{*+}} &= (0.81 \pm 0.11) \text{ MeV} . \end{aligned} \tag{5.2}$$

The only exception is the D^* case, where one observes tension between these two sets of the results; notice, however, that also the uncertainties of these predictions are rather large.

Comparing with available lattice-QCD results on the IB, our results agree nicely for the D -mesons ($f_{D^+} - f_{D^0} = 0.94^{+0.5}_{-0.12}$ MeV [22]), but have a very strong tension for the B -mesons ($f_{B^0} - f_{B^+} = 3.8 \pm 1.0$ MeV [22]).

It should be emphasized that the present approach based on the combination of OPE and a few inputs from lattice QCD potentially has fewer theoretical uncertainties than other formulations of QCD sum rules: first, the condensate contributions, in particular, those of the quark condensate, which produced the main OPE error in the decay constants, vanish in the LD limit; second, the systematic uncertainty of the sum-rule method is now encoded in only one quantity — the effective threshold, which may be fixed to good accuracy due to the use of the few accurate lattice inputs.

In conclusion, QCD sum rules for the mass dimension-2 Borelized invariant amplitudes at $\tau = 0$ (i.e., an infinitely large Borel mass parameter) provide an efficient tool for the analysis of the dependence of decay constants (and potentially of other hadron observables) on quark masses.

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Appendix A: Isospin breaking in the OPE

The two-point Green function Π of interest is given by the functional integral

$$\langle T(j_1(y)j_2(0)) \rangle = \frac{\int D\psi(x)D\bar{\psi}(x)DA_\mu(x) j_1(y)j_2(0) \exp(i \int d^4x L(x))}{\int D\psi(x)D\bar{\psi}(x)DA_\mu(x) \exp(i \int d^4x L(x))}, \quad (\text{A.1})$$

where j_1 and j_2 are (gauge-invariant) operators constructed from quark and gluon fields, and

$$L(x) = L^{(0)}(x) - \frac{1}{2}\delta m \bar{q}(x)q(x), \quad \bar{q}(x)q(x) \equiv \bar{d}(x)d(x) - \bar{u}(x)u(x), \quad \delta m \equiv m_d - m_u. \quad (\text{A.2})$$

Here, $L^{(0)}(x)$ is the $SU(2)$ -symmetric Lagrangian describing two equal-mass quarks, with the quark-mass term

$$m(\bar{d}d + \bar{u}u), \quad m \equiv \frac{1}{2}(m_d + m_u). \quad (\text{A.3})$$

Important for our argument is that the operators j_1 and j_2 do not contain light-quark masses explicitly, although they, of course, contain the light-quark field operators. For instance, one may consider

$$j_1(x) = \bar{q}(x)\gamma_\mu\gamma_5 Q(x), \quad j_2(x) = (j_1(x))^\dagger = \bar{Q}(x)\gamma_\mu\gamma_5 q(x), \quad q = u, d. \quad (\text{A.4})$$

Expanding (A.1) in powers of δm , one finds

$$i\langle T(j_1(y)j_2(0)) \rangle = i\langle T(j_1(y)j_2(0)) \rangle^{(0)} - \frac{\delta m}{2}i^2 \left\langle T \left(j_1(y)j_2(0) \int \bar{q}(z)q(z)dz \right) \right\rangle^{(0)} + O(\delta m^2), \quad (\text{A.5})$$

the superscript “0” indicating that the full Green functions correspond to $SU(2)$ -symmetric QCD with two light quarks of the same mass m . Let us emphasize the appealing feature of the expansion (A.5): at each order in δm , one encounters the full Green function of the $SU(2)$ -symmetric QCD.

One may expect that the power corrections in the OPE for the three-point Green function Γ of the $SU(2)$ -symmetric QCD are the $SU(2)$ -symmetric condensates, e.g., $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle$. However, in order to obtain the contributions of interest, we need to perform the limit $q \rightarrow 0$. This step cannot be done easily: the OPE for Γ is only given by local $SU(2)$ -symmetric condensates if one keeps q^2 large and negative; a straightforward extension of the known power corrections to $q \rightarrow 0$ leads to a wrong result: it is known that if one naively extends power corrections in the vector three-point function to $q = 0$, then they do not satisfy the Ward identity (see, e.g., [25]).

On the other hand, one can proceed by expanding $\Pi(p)$ in powers of the small quark mass; then the mass derivatives emerge. Translating the expression (A.5) into momentum space, to $O(\delta m)$ accuracy we obtain

$$\Pi(p) = \Pi^{(0)}(p) - \frac{1}{2}\delta m \Gamma^{(0)}(p, q = 0), \quad (\text{A.6})$$

where $\Pi^{(0)}(p)$ is the full two-point function of $SU(2)$ -symmetric QCD, and $\Gamma^{(0)}(p, q = 0)$ is the three-point function of the scalar current $\bar{q}q$ at zero momentum transfer, also calculated in the full $SU(2)$ -symmetric theory. Consequently, finding the leading-order $SU(2)$ -breaking effects reduces to calculating the Green functions in $SU(2)$ -symmetric QCD.

Now, consider the correlation functions

$$\Pi(p) = i \int d^4y \exp(ipy) \langle T(j_1(y)j_2(0)) \rangle, \quad \Gamma(p, q) = i^2 \int d^4y d^4z \exp(ipy) \exp(-iqz) \langle T(j_1(y)j_2(0)\bar{q}(z)q(z)) \rangle. \quad (\text{A.7})$$

For the two-point function, we can write the dispersion representation

$$\Pi^{(0)}(p) = \int_{(m_Q+m)^2}^{\infty} \frac{ds}{s-p^2} \rho(s, m_Q, m) \theta(s - (m_Q + m))^2. \quad (\text{A.8})$$

Using the well-known relation

$$\Gamma^{(0)}(p, q = 0) = -\frac{\partial}{\partial m} \Pi^{(0)}(p), \quad (\text{A.9})$$

the three-point function at zero momentum transfer may be related to the mass derivative of the two-point function, which then leads to the appearance of the mass derivatives of the quark condensate.

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